Dust acoustic shock waves

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It is shown that nonlinear equations governing the dynamics of large amplitude nondispersive dust acoustic waves admit nonstationary dust acoustic shock waves. Analytical and numerical results for the latter are presented, and the relevance of our investigation to laboratory experiments is discussed.

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In their classic paper, Rao, Shukla, and Yu [1] predicted linear and nonlinear properties of dust acoustic waves (DAW's) in an unmagnetized dusty plasma. The DAW's are low phase speed (in comparison with the electron and ion thermal speeds) electrostatic oscillations supported by the restoring force coming from the inertialess electron and ion fluids, as well as the inertia of charged dust grains which are a billion times heavier than ions. Thus the DAW's appear on a kinetic level, which have been visualized by the naked eye in several laboratory experiments [2–7]. In radio-frequency dusty plasma discharges, typical wave frequencies (wavelengths) are 10 Hz (half a centimeter) while the wave phase speed is roughly 5 cm/s. Typical images of the DAW's reveal that waves are of large amplitudes and their wave fronts are steepened. Thus nonlinearity is in action. It has been found that harmonic generation nonlinearity can give rise to small amplitude acoustic dust acoustic solitary waves, which are characterized by an inverted bell shaped potential distribution, obeying a Korteweg–de Vries equation [2,8]. Arbitrary large amplitude dust acoustic solitary waves are shown to exist in the steady state only [9,10].

In this Brief Report, we present *nonstationary solutions* of fully nonlinear nondispersive DAW's in an unmagnetized dusty plasma. Since the phase speed of the DAW's is much smaller than the electron and ion thermal speeds, inertialess electrons and ions are in thermal equilibrium in the dust acoustic wave potential ϕ. Accordingly, we have for the electron and ion number densities, respectively,

$$n_e(\phi) = n_{e0} \exp(e\phi/T_e), \tag{1}$$

and

$$n_i(\phi) = n_{i0} \exp(-e\phi/T_i), \tag{2}$$

where n_{e0} (n_{i0}) is the equilibrium electron (ion) number density, e is the magnitude of the electron charge, and T_e (T_i) is the electron (ion) temperature. At equilibrium, we have $n_{i0} = n_{e0} - \epsilon Z_d n_{d0}$, where n_{d0} is the unperturbed dust number density, Z_d is the dust charge state, and ϵ equals -1 (+1) for negatively (positively) charged dust grains.

The dynamics of dust grains is governed by

$$\frac{\partial n_d}{\partial t} + \nabla \cdot (n_d \mathbf{v}_d) = 0, \tag{3}$$

and

$$\frac{\partial \mathbf{v}_d}{\partial t} + \mathbf{v}_d \cdot \nabla \mathbf{v}_d = -\epsilon \frac{Z_d e}{m_d} \nabla \phi - \frac{3 n_d T_d}{n_{d0}^2 m_d} \nabla n_d, \tag{4}$$

where n_d is the dust number density, $\mathbf{v}_d$ is the dust fluid velocity, and T_d is the dust temperature.

The equations are closed by means of Poisson's equation,

$$\nabla^2 \phi = 4\pi e (n_e - n_i - \epsilon Z_d n_d). \tag{5}$$

In the following, we study nonstationary properties of one-dimensional nondispersive fully nonlinear DAW's. Thus, from $n_e - n_i - \epsilon Z_d n_d = 0$, we have

$$N_d = [\alpha \exp(-\tau\varphi) - \exp(\varphi)]/(\alpha - 1), \tag{6}$$

where $N_d = n_d/n_{d0}$, $\alpha = n_{i0}/n_{e0}$, $\tau = T_e/T_i$, and $\varphi = e\phi/T_e$.

By using Eq. (6) we can then write Eqs. (3) and (4) as

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)[\alpha \exp(-\tau\varphi) - \exp(\varphi)] + [\alpha \exp(-\tau\varphi) - \exp(\varphi)]\frac{\partial u}{\partial x} = 0, \tag{7}$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\epsilon \frac{\partial \varphi}{\partial x} - \frac{3T_d}{2 Z_d T_e (\alpha - 1)^2}\frac{\partial}{\partial x} \times [\alpha \exp(-\tau\varphi) - \exp(\varphi)]^2, \tag{8}$$

where u is the normalized [by $C_d = (Z_d T_e/m_d)^{1/2}$] dust fluid velocity along the x axis, the time and space are normalized by the dust plasma period ω_{pd}^{-1} and the Debye radius C_d/ω_{pd}, respectively, where $\omega_{pd} = (4\pi Z_d^2 e^2 n_{d0}/m_d)^{1/2}$ is the dust plasma frequency. In most situations, the dust temperature effect can be neglected due to the low dust temperature T_d and large dust charge state Z_d. In the following, we will therefore neglect the last term in Eq. (8). Equations (7) and (8) form a pair for investigating dust acoustic shock waves in dusty plasmas.

Similar to acoustic waves in air [11] or ion acoustic waves in plasmas [12], the nonlinear DA waves may develop shocklike steepened profiles in due course of time. A special class of solutions to Eqs. (7) and (8) are simple-wave solutions, in which the creation of shock waves can be more easily studied than in the original system. Carrying out the x

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and t differentiation in the first term of Eq. (7), we can rewrite Eqs. (7) and (8) in the matrix form [neglecting the last term in Eq. (8)] as

$$\frac{\partial}{\partial t}\begin{bmatrix}\varphi\\ u\end{bmatrix}+\begin{bmatrix}u & -\chi(\varphi)\\ \epsilon & u\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}\varphi\\ u\end{bmatrix}=0, \tag{9}$$

where $\chi(\varphi)=[\alpha\exp(-\tau\varphi)-\exp(\varphi)]/[\alpha\tau\exp(-\tau\varphi)+\exp(\varphi)]$. The square matrix on the left-hand side of Eq. (9) is diagonalized by means of a diagonalizing matrix consisting of the eigenvectors to the matrix in Eq. (9). The two eigenvalues to the matrix are

$$\lambda_\pm = u \pm \sqrt{-\epsilon\chi(\varphi)}, \tag{10}$$

and a diagonalizing matrix, where the columns are the eigenvectors, is

$$C=\begin{bmatrix}1 & 1\\ -\sqrt{-\epsilon/\chi} & \sqrt{-\epsilon/\chi}\end{bmatrix}. \tag{11}$$

Carrying out the algebra, the system of equations (9) is diagonalized as

$$\frac{\partial\psi_+}{\partial t}+\lambda_+\frac{\partial\psi_+}{\partial x}=0, \tag{12}$$

$$\frac{\partial\psi_-}{\partial t}+\lambda_-\frac{\partial\psi_-}{\partial x}=0, \tag{13}$$

where $\psi_\pm=u\mp F(\varphi)$ and $F(\varphi)=\int_0^\varphi[-\epsilon/\chi(s)]^{1/2}\,ds$. A simple-wave solution is found by setting either ψ_+ or ψ_- to zero. Setting ψ_- to zero, we obtain $u=-F(\varphi)$, $\psi_+=2u$, and from Eq. (12)

$$\frac{\partial u}{\partial t}+\lambda_+(\varphi)\frac{\partial u}{\partial x}=0, \tag{14}$$

where $\lambda_+(\varphi)=-F(\varphi)+\sqrt{-\epsilon\chi(\varphi)}$. Since u is a function of φ, we also have an equation similar to Eq. (14) for φ, i.e.,

$$\frac{\partial\varphi}{\partial t}+\lambda_+(\varphi)\frac{\partial\varphi}{\partial x}=0, \tag{15}$$

which describes the self-steepening of the potential φ. The general solution of Eq. (15), before any shocks have developed, is $\varphi=f_0[x-\lambda_+(\varphi)t]$, where f_0 is a function of one variable, given as an initial condition at $t=0$. The effective phase speed $\lambda_+(\varphi)$ is a function of the solution φ, and similar to the inviscid Burger's equation, the general solution may therefore self-steepen and develop shocks after some time. After shocks have developed, their details depend strongly on the dispersive properties of the DA waves on small scale lengths, but the shock fronts typically propagate with speeds according to the Rankine-Hugoniot condition, $v_{shock}=[\Lambda(\varphi_{left})-\Lambda(\varphi_{right})]/(\varphi_{left}-\varphi_{right})$, where φ_{left} and φ_{right} are the values of φ to the left and right of the shock, respectively, and the flow function Λ is a primitive function of $\lambda_+(\varphi)$.

Approximative solutions can be obtained in the small-amplitude limit, viz. $|\varphi|,\tau|\varphi|\ll 1$. A first-order Taylor expansion of $\lambda_+(\varphi)$ with respect to φ gives $\lambda_+(\varphi)=c_0-\beta\varphi$, where the linear dust acoustic speed (normalized by C_d) is $c_0=\sqrt{-\epsilon\chi(0)}=\sqrt{-\epsilon(\alpha-1)/(\tau\alpha+1)}$ and $\beta=[-\epsilon(\alpha\tau+1)/(\alpha-1)]^{1/2}[1+\alpha(1+\alpha)^2/2(1+\alpha\tau)^2]$ accounts for the first-order nonlinear modification of the wave speed. Specifically, for $\epsilon=-1$ and for negative φ, the wave speed increases with larger negative values of φ, giving rise to self-steepening and shocks. When shock fronts have developed, the speed (normalized to C_d) of the shock front is $v_{shock}=c_0-\beta(\varphi_{left}+\varphi_{right})/2$. A Taylor expansion of Eq. (6) with respect to φ gives the small-amplitude relation $\varphi=(1-N_d)(\alpha-1)/(\alpha\tau+1)$, which may be used to express the effective wave and shock speeds in terms of the dust density N_d. The result is $v_{ph}=\lambda_+(\varphi)=c_0\{1+(N_d-1)[1+\alpha(1+\alpha)^2/2(1+\alpha\tau)^2]\}\approx c_0N_d$ if $\tau\gg 1$ and $v_{shock}=c_0\{1+0.5(N_{d,left}+N_{d,right}-2)[1+\alpha(1+\alpha)^2/2(1+\alpha\tau)^2]\}\approx c_0(N_{d,left}+N_{d,right})/2$, which show the nonlinear modification of the phase speed compared to the linear phase speed c_0. Since the phase speed v_{ph} is proportional to the dust density N_d, shocks may be created because dust density maxima will travel faster than dust density minima.

We now compare our theory with recent experiments (described in connection with Figs. 4–6 in Ref. [6]), where it was observed that large-amplitude waves, $N_d=2.2$, were associated with a ratio $u/v_{ph}\sim 0.5$–0.8 between the maximum particle particle (fluid) velocities and the phase speed of the waves. Linear theory gives $v_{ph}/u=N_d-1=1.2$, which deviates from the experimental results [6]. Hence we compare the experimental values with our theory. Using the dust density $N_d=2.2$, the theoretical ratio between the particle and phase velocities, before shocks have developed, is $u/v_{ph}=(N_d-1)/N_d\approx 0.55$, and the ratio between the particle velocity and the speed of fully developed shock fronts is $u/v_{shock}=2(N_d-1)/(N_{d,left}-N_{d,right})\approx 0.75$, where we used $N_{d,left}=2.2$ and $N_{d,right}=1$ ($N_{d,right}$ is the unperturbed dust density in front of the shock). The theoretical values are thus in excellent agreement with the experimental ones.

In order to study the temporal development localized dust acoustic shock waves and to compare them with experiments [6], we have solved the simple-wave equation (15) numerically. We have chosen the parameters of the laboratory experiment [6], where silica (SiO_2) dust grains, with the diameter ≈ 3 μm and mass 3.5×10^{-14} kg, were used. Further, we used the dust charge $Z_d=10^5$, the electron temperature $T_e=2.5$ eV , the dust number density $n_d=10^9$ m^{-1}, $\tau=100$, and $\alpha=1.1$. For these parameters, we have $C_d=1.1$ m/s, $\omega_{pd}^{-1}=1.1\times 10^{-3}$ s, and $C_d/\omega_{pd}=1.2\times 10^{-3}$ m. Initially, the potential was set to a localized pulse, $\varphi=-10^{-3}\,\mathrm{sech}(0.1x)$. The results are displayed in dimensional units in Fig. 1. In order to convert the dimensionless units to dimensional ones used in the experiment, the spatial values were multiplied by 1.2 in order to obtain the position in millimeters, the time values were multiplied by 1.1×10^{-3} in order to obtain values in seconds, and the velocity values u were multiplied by 1200 in order to obtain values in mm/s. The values of the potential φ were multiplied by 2.5 to convert them to V.

The initial pulse is the curve labeled (a) in Fig. 1, with its negative potential (upper panel), dust velocity (middle panel) and dust density (lower panel). We observe that the maximum density ≈ 2.2 (lower panel) is correlated with a fluid

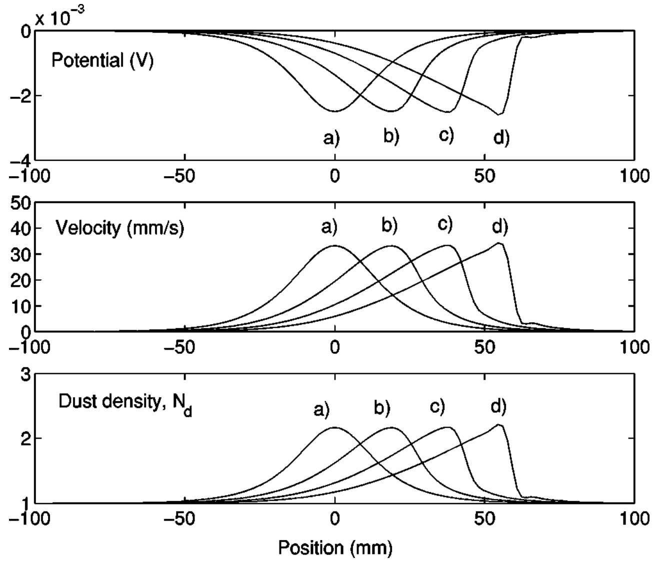

FIG. 1. The time evolution of the electrostatic potential (upper panel), dust velocity (middle panel), and dust density (lower panel) with $\alpha=1.1$ and $\tau=100$, for (a) $t=0$ s, (b) $t=0.24$ s, (c) $t=0.48$ s, and (d) $t=0.73$ s.

(particle) velocity of approximately 35 mm/s; see the middle panel, which agrees relatively well with the values observed in the experiment (see the discussion in connection with Figs. 5 and 6 in Ref. [6]). We see a clear signature of self-steepening of the DAW's and the creation of shocks [curves marked (d) in Fig. 1], which can also be seen in some of the experiments of Ref. [6]. In Fig. 1, the maximum phase speed of the wave can be calculated from the change of position $\Delta x \approx 40$ mm of the density maximum in curve (c) compared to curve (a), divided by the time $\Delta t = 0.48$ s, to be $v_{ph} \approx 83$ mm/s, i.e., somewhat more than twice the maximum fluid velocity, shown in the middle panel (in the experiment and in the low-amplitude approximation discussed above, the phase velocity was somewhat less than twice the maximum fluid velocity).

To summarize, we have considered the nonstationary propagation of fully nonlinear nondispersive DAW's in an unmagnetized dusty plasma. We have employed Boltzmann electron and ion distributions and the hydrodynamic equations for the dust fluid to derive a set of wave characteristic equations which are then solved numerically. We have derived simple nonlinear relations between the DAW density amplitude and its phase and fluid velocities, which explain observations in experiments that cannot be explained by a linear theory. Our numerical results reveal dust acoustic wave breaking and the formation of finite amplitude dust acoustic shock waves due to nonlinear effects. Dust acoustic shock waves seem to be observed in experiments [6].

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